



Why

Let $A \in \mathbf{R}^{m \times n}$ and define $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ by $f(x) = Ax$. Then f is a linear function from \mathbf{R}^n to \mathbf{R}^m . Conversely, suppose $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a linear function. Then there exists a matrix $B \in \mathbf{R}^{m \times n}$ so that $g(z) = Bz$. Does this function have an inverse?

Derivation

If $A \in \mathbf{R}^{m \times n}$, with $m \neq n$, then the inverse of f can not exist. For a square matrix $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times n}$ is a *left inverse* if $BA = I$. In other words, B is a left inverse element of A in the algebra of matrices with the operation of multiplication. $C \in \mathbf{R}^{n \times n}$ is a *right inverse* if $AC = I$.

Definition

We call a square matrix A *invertible* if there is $B \in \mathbf{R}^{n \times n}$ so that $BA = I$.

Now suppose that $A \in \mathbf{R}^{n \times n}$. Of course, the inverse may not exist. Consider, for example if A was the n by n matrix of zeros. If there exists a matrix B so that $BA = I$ we call B the *left inverse* of A and likewise if $AC = I$ we call C the *right inverse* of A . In the case that A is square, the right inverse and left inverse coincide.

Proposition 1. *Suppose $A, B, C \in \mathbf{R}^{n \times n}$. If $BA = I$ and $AC = I$, then $B = C$.*

Proof. Since $BA = AC$ we have $BBA = BAC$ so $B = C$ since $BA = I$. □

