



## Why

What are the affine sets in terms of subspaces?

### Affine sets which are subspaces

The subspaces of  $\mathbf{R}^n$  are the affine sets which contain the origin.

**Proposition 1.**  $M \subset \mathbf{R}^n$  is a subspace if and only if  $M$  is affine and  $0 \in M$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $M$  is a subspace. Then  $0 \in M$ . Also  $\alpha x + \beta y \in M$  for all  $\alpha, \beta \in \mathbf{R}$  and  $x, y \in \mathbf{R}^n$ . In particular,  $(1 - \lambda)x + \lambda y \in M$  for all  $\lambda \in \mathbf{R}$ ,  $x, y \in \mathbf{R}^n$ . In other words,  $M$  contains the line through  $x$  and  $y$ .

( $\Leftarrow$ ) Suppose  $M$  is affine and  $0 \in M$ .  $M$  is closed under scalar multiplication since

$$\alpha x = (1 - \alpha)0 + \alpha x$$

is in the line through  $0$  and  $x$ .  $M$  is closed under vector addition since

$$(1/2)(x + y) = (1 - 1/2)x + (1/2)y$$

is in the line through  $x$  and  $y$ . Thus,  $x + y = 2(1/2)(x + y) \in M$ .  $\square$

### Affine sets as translated subspaces

**Proposition 2.** Suppose  $M \neq \emptyset$  is affine. Then there exists a unique subspace  $L$  and vector  $a \in \mathbf{R}^n$  for which  $M = L + a$ . Moreover,

$$L = M - M = \{x - y \mid x, y \in M\}.$$

*Proof.* First, uniqueness. Suppose  $L_1$  and  $L_2$  are subspaces parallel to  $M$ . We will show that  $L_2 \supset L_1$  (and similarly,  $L_1 \supset L_2$ ).

Since  $L_1$  and  $L_2$  are both parallel to  $M$ , they are also parallel to each other. Consequently, there exists  $a \in \mathbf{R}^n$  with  $L_2 = L_1 + a$ . Since  $0 \in L_2$  (it is a subspace, after all),  $-a \in L_1$ . Since  $L_1$  is a subspace,  $a \in L_1$ .

So  $x + a \in L_1$  for every  $a \in L_1$ , and so  $L_2 = L_1 + a \subset L_1$ . A similar argument gives  $L_1 \supset L_2$ .

If  $y \in M$ , then  $M + (-y) = M - y$  is a translate of  $M$  containing zero (since  $y - y = 0$ ). In other words, the affine set  $M - y$  is a subspace. This, then, is the unique subspace parallel to  $M$ . Since  $y$  was arbitrary, the subspace parallel to  $M$  is  $L = \cup_{y \in M} M - y = \{x - y \mid x, y \in M\}$ .  $\square$

