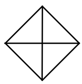


Why

In ordinary reduction, we obtain a sequence of row reducers.

Factorization of A from a sequence of reducers

Let $(A \in \mathbf{R}^{m \times m}, b \in \mathbf{R}^m)$ be an ordinarily reducible linear system. The *ordinary reducer sequence* is a sequence of reducer matrices $L_1, \dots L_{m-1}$ with $A_1 = L_1 A$ and $A_i = L_i A_{i-1}$ for $2 \leq i \leq m-1$. In other words, $U \in \mathbf{R}^{m \times m}$ defined by

$$U = L_{m-1} \cdots L_2 L_1 A \tag{1}$$

is the ordinary row reduction of A. U is upper triangular.

If $L_{m-1} \cdot s L_2 L_1$ in Equation (1) is invertible, then we have

$$A = (L_{m-1} \cdot s L_2 L_1)^{-1} U,$$

which is a factorization of A. Each L_i is invertible, so

$$(L_{m-1} \cdots L_2 L_1)^{-1} = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}.$$

So we are interested in the inverse of L_i for $i \leq m-1$. Recall that if x_1 is the first column of A, and x_2 is the second column of $L_1 A$ and x_k is the kth column of $L_{k-1} \cdot s L_1 A$ for $k = 2, \dots, m-1$, then

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -\ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -\ell_{mk} & & & 1 \end{bmatrix}$$

where $\ell_{jk} = x_{jk}/x_{kk}$ for $k < j \leq m$.

Properties

The two important properties of the L_i is that they have simple inverses and a simple product. Define

$$\ell_k = (0, \cdot s, 0, \ell_{k+1,k}, \cdot s, \ell_{m,k})$$

so that $L_k = L_k - \ell_k e_k^\top$ where $(e_k)_i$ is 1 if $k = i$ and 0 otherwise.

Proposition 1. *L_i^{-1} is L_i with the subdiagonal entries negated.*

Proof. From the sparsity pattern of ℓ_k, we have $e_k^\top \ell_k = 0$. So

$$(I - \ell_k e_k^\top)(I + \ell_k e_k^\top) = I - \ell_k e_k^\top \ell_k e_k^\top = I.$$

□

Proposition 2. *$L_k^{-1} L_{k+1}^{-1}$ is the unit lower-triangular matrix with the entries of both L_k^{-1} and L_{k+1}^{-1} in their usual places.*

Proof. From the sparsity pattern of ℓ_{k+1} we have $e_k^\top \ell_{k+1} = 0$ so that

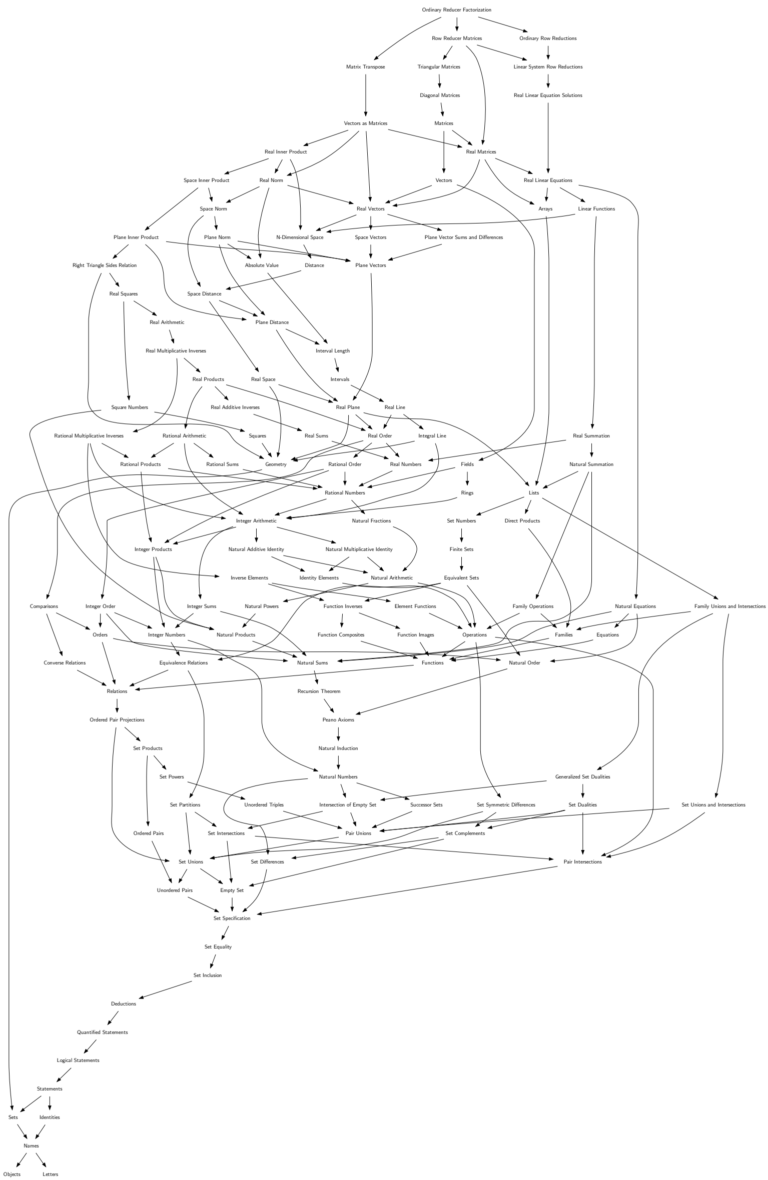
$$L_k^{-1} L_{k+1}^{-1} = (I + \ell_k e_k^\top)(I + \ell_{k+1} e_{k+1}^\top) = I + \ell_k e_k^* + \ell_{k+1} e_{k+1}^*.$$

□

Combining these two results, we deduce that

$$L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1} = \begin{bmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{m1} & \ell_{m2} & \cdot s & \ell_{m,m-1} & 1 \end{bmatrix}$$

If we define $L = L_1^{-1} \cdots L_{m-1}^{-1}$ we obtain $A = LU$. In other words, we have a factorization (the *ordinary reducer factorization*) of A in terms of two matrices. The first, L is unit lower triangular. The second, U, is upper triangular.


Ordinary Reducer Factorization
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Real Arithmetic
Plane Distance
Real Multiplicative Inverses
Interval Length
Real Products
Real Space
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Real Additive Inverses
Real Plane
Real Line
Rational Multiplicative Inverses
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Natural Multiplicative Identity
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Inverse Elements
Identity Elements
Natural Arithmetic
Equivalent Sets
Comparisons
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Integer Sums
Natural Powers
Function Inverses
Element Functions
Family Operations
Natural Equations
Family Unions and Intersections
Orders
Integer Numbers
Natural Products
Function Composites
Function Images
Operations
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Natural Induction
Set Powers
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Generalized Set Dualities
Set Partitions
Unordered Triples
Intersection of Empty Set
Successor Sets
Set Symmetric Differences
Set Dualities
Set Unions and Intersections
Ordered Pairs
Set Intersections
Pair Unions
Set Complements
Set Unions
Set Differences
Pair Intersections
Unordered Pairs
Empty Set
Set Specification
Set Equality
Set Inclusion
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